

# Interval cyclic edge-colorings of graphs

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A proper edge-coloring of a graph  $G$  with colors  $1, \dots, t$  is called an *interval cyclic  $t$ -coloring* if all colors are used, and the edges incident to each vertex  $v \in V(G)$  are colored by  $d_G(v)$  consecutive colors modulo  $t$ , where  $d_G(v)$  is the degree of a vertex  $v$  in  $G$ . A graph  $G$  is *interval cyclically colorable* if it has an interval cyclic  $t$ -coloring for some positive integer  $t$ . The set of all interval cyclically colorable graphs is denoted by  $\mathfrak{N}_c$ . For a graph  $G \in \mathfrak{N}_c$ , the least and the greatest values of  $t$  for which it has an interval cyclic  $t$ -coloring are denoted by  $w_c(G)$  and  $W_c(G)$ , respectively. In this paper we investigate some properties of interval cyclic colorings. In particular, we prove that if  $G$  is a triangle-free graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq |V(G)| + \Delta(G) - 2$ . We also obtain bounds on  $w_c(G)$  and  $W_c(G)$  for various classes of graphs. Finally, we give some methods for constructing of interval cyclically non-colorable graphs.

Keywords: edge-coloring, interval coloring, interval cyclic coloring, bipartite graph, complete graph.

## 1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let  $V(G)$  and  $E(G)$  denote the sets of vertices and edges of  $G$ , respectively. For a graph  $G$ , the number of connected components of  $G$  is denoted by  $c(G)$ . A graph  $G$  is Eulerian if it has a closed trail containing every edge of  $G$ . The degree of a vertex  $v \in V(G)$  is denoted by  $d_G(v)$  (or  $d(v)$ ), the maximum degree of  $G$  by  $\Delta(G)$ , the diameter of  $G$  by  $\text{diam}(G)$ , and the chromatic index of  $G$  by  $\chi'(G)$ . The terms and concepts that we do not define can be found in [3, 33].

A proper edge-coloring of a graph  $G$  with colors  $1, \dots, t$  is an *interval  $t$ -coloring* if all colors are used, and the colors of edges incident to each vertex of  $G$  are form an interval of integers. A graph  $G$  is *interval colorable* if it has an interval  $t$ -coloring for some positive integer  $t$ . The concept of interval edge-coloring of graphs was introduced by Asratian

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and Kamalian [1]. In [1, 2], the authors showed that if  $G$  is interval colorable, then  $\chi'(G) = \Delta(G)$ . In [1, 2], they also proved that if a triangle-free graph  $G$  has an interval  $t$ -coloring, then  $t \leq |V(G)| - 1$ . Later, Kamalian [13] showed that if  $G$  admits an interval  $t$ -coloring, then  $t \leq 2|V(G)| - 3$ . This upper bound was improved to  $2|V(G)| - 4$  for graphs  $G$  with at least three vertices [8]. For an  $r$ -regular graph  $G$ , Kamalian and Petrosyan [18] showed that if  $G$  with at least  $2r + 2$  vertices admits an interval  $t$ -coloring, then  $t \leq 2|V(G)| - 5$ . For a planar graph  $G$ , Axenovich [4] showed that if  $G$  has an interval  $t$ -coloring, then  $t \leq \frac{11}{6}|V(G)|$ . In [12, 13, 24, 26], interval colorings of complete graphs, complete bipartite graphs, trees, and  $n$ -dimensional cubes were investigated. The  $NP$ -completeness of the problem of the existence of an interval coloring of an arbitrary bipartite graph was shown in [29]. In [5, 6, 21, 25, 26, 27], interval colorings of various products of graphs were investigated. In [2, 3, 6, 7, 9, 10, 11, 16, 17, 19], the problem of the existence and construction of interval colorings was considered, and some bounds for the number of colors in such colorings of graphs were given.

A proper edge-coloring of a graph  $G$  with colors  $1, \dots, t$  is called an *interval cyclic  $t$ -coloring* if all colors are used, and the edges incident to each vertex  $v \in V(G)$  are colored by  $d_G(v)$  consecutive colors modulo  $t$ . A graph  $G$  is *interval cyclically colorable* if it has an interval cyclic  $t$ -coloring for some positive integer  $t$ . This type of edge-coloring under the name of “ $\pi$ -coloring” was first considered by Kotzig in [20], where he proved that every cubic graph has a  $\pi$ -coloring with 5 colors. However, the concept of interval cyclic edge-coloring of graphs was explicitly introduced by de Werra and Solot [32]. In [32], they proved that if  $G$  is an outerplanar bipartite graph, then  $G$  has an interval cyclic  $t$ -coloring for any  $t \geq \Delta(G)$ . In [22], Kubale and Nadolski showed that the problem of determining whether a given bipartite graph is interval cyclically colorable is  $NP$ -complete. Later, Nadolski [23] showed that if  $G$  is interval colorable, then  $G$  has an interval cyclic  $\Delta(G)$ -coloring. He also proved that if  $G$  is a connected graph with  $\Delta(G) = 3$ , then  $G$  has an interval cyclic coloring with at most 4 colors. In [14, 15], Kamalian investigated interval cyclic colorings of simple cycles and trees. For simple cycles and trees, he determined all possible values of  $t$  for which these graphs have an interval cyclic  $t$ -coloring.

In this paper we investigate some properties of interval cyclic colorings. In particular, we prove that if a triangle-free graph  $G$  with at least two vertices has an interval cyclic  $t$ -coloring, then  $t \leq |V(G)| + \Delta(G) - 2$ . For various classes of graphs, we also obtain bounds on the least and the greatest values of  $t$  for which these graphs have an interval cyclic  $t$ -coloring. Finally, we describe some methods for constructing of interval cyclically non-colorable graphs.

## 2. Notations, definitions and auxiliary results

We use standard notations  $C_n, K_n$  and  $Q_n$  for the simple cycle, complete graph on  $n$  vertices and the hypercube, respectively. We also use standard notations  $K_{m,n}$  and  $K_{l,m,n}$  for the complete bipartite and tripartite graph, respectively, one part of which has  $m$  vertices, other part has  $n$  vertices and a third part has  $l$  vertices.

A *partial edge-coloring* of  $G$  is a coloring of some of the edges of  $G$  such that no two adjacent edges receive the same color. If  $\alpha$  is a partial edge-coloring of  $G$  and  $v \in V(G)$ ,

then  $S(v, \alpha)$  denotes the set of colors appearing on colored edges incident to  $v$ .

A graph  $G$  is *interval colorable* if it has an interval  $t$ -coloring for some positive integer  $t$ . The set of all interval colorable graphs is denoted by  $\mathfrak{N}$ . For a graph  $G \in \mathfrak{N}$ , the least and the greatest values of  $t$  for which it has an interval  $t$ -coloring are denoted by  $w(G)$  and  $W(G)$ , respectively.

A graph  $G$  is *interval cyclically colorable* if it has an interval cyclic  $t$ -coloring for some positive integer  $t$ . The set of all interval cyclically colorable graphs is denoted by  $\mathfrak{N}_c$ . For a graph  $G \in \mathfrak{N}_c$ , the least and the greatest values of  $t$  for which it has an interval cyclic  $t$ -coloring are denoted by  $w_c(G)$  and  $W_c(G)$ , respectively. The *feasible set*  $F(G)$  of a graph  $G$  is the set of all  $t$ 's such that there exists an interval cyclic  $t$ -coloring of  $G$ . The feasible set of  $G$  is *gap-free* if  $F(G) = [w_c(G), W_c(G)]$ . Clearly, if  $G \in \mathfrak{N}$ , then  $G \in \mathfrak{N}_c$  and  $\chi'(G) \leq w_c(G) \leq w(G) \leq W(G) \leq W_c(G) \leq |E(G)|$ .

Let  $\lfloor a \rfloor$  denote the largest integer less than or equal to  $a$ . For two positive integers  $a$  and  $b$  with  $a \leq b$ , we denote by  $[a, b]$  the interval of integers  $\{a, \dots, b\}$ . By  $[a, b]_{\text{even}}$  ( $[a, b]_{\text{odd}}$ ), we denote the set of all even (odd) numbers from the interval  $[a, b]$ .

In [1, 2], Asratian and Kamalian obtained the following two results.

**Theorem 1** *If  $G \in \mathfrak{N}$ , then  $\chi'(G) = \Delta(G)$ . Moreover, if  $G$  is a regular graph, then  $G \in \mathfrak{N}$  if and only if  $\chi'(G) = \Delta(G)$ .*

**Theorem 2** *If  $G$  is a connected triangle-free graph and  $G \in \mathfrak{N}$ , then*

$$W(G) \leq |V(G)| - 1.$$

For general graphs, Kamalian proved the following

**Theorem 3** ([13]). *If  $G$  is a connected graph with at least two vertices and  $G \in \mathfrak{N}$ , then*

$$W(G) \leq 2|V(G)| - 3.$$

Note that the upper bound in Theorem 3 is sharp for  $K_2$ , but if  $G \neq K_2$ , then this upper bound can be improved.

**Theorem 4** ([8]). *If  $G$  is a connected graph with at least three vertices and  $G \in \mathfrak{N}$ , then*

$$W(G) \leq 2|V(G)| - 4.$$

In [30], Vizing proved the following well-known result.

**Theorem 5** *For every graph  $G$ ,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

**Corollary 6** *If  $G$  is a regular graph, then  $G \in \mathfrak{N}_c$  and  $w_c(G) = \chi'(G)$ .*

From Theorems 1 and 5, we get

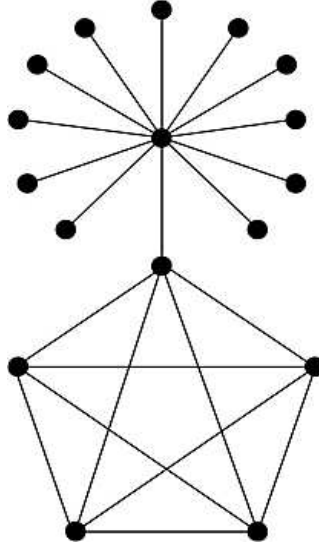


Figure 1. The interval cyclically non-colorable graph.

**Corollary 7**  $\mathfrak{N} \subset \mathfrak{N}_c$ .

Although all regular graphs are interval cyclically colorable, there are many graphs that have no interval cyclic coloring. In Fig. 1, we present the smallest known interval cyclically non-colorable graph.

We also need the generalizations of Theorems 2,3 and 4 for disconnected graphs. It can be easily proved by induction on the number of connected components that the following two lemmas hold.

**Lemma 8** *If  $G$  is a triangle-free graph and  $G \in \mathfrak{N}_c$ , then*

$$W(G) \leq |V(G)| - c(G).$$

**Lemma 9** *If  $G$  is a graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then*

$$W(G) \leq 2|V(G)| - 3 \cdot c(G).$$

*Moreover, if  $G$  has at least three vertices, then*

$$W(G) \leq 2|V(G)| - 4 \cdot c(G).$$

### 3. Some general results

In this section we derive some upper bounds for  $W_c(G)$  depending on the number of vertices, degrees and diameter for connected graphs, triangle-free graphs, and, in particular, for bipartite graphs. Next we show that there are graphs  $G$  for which  $w_c(G) > \chi'(G)$ . We also investigate the feasible sets of interval cyclically colorable graphs. In particular, we show that if  $G$  is interval colorable, then  $[\Delta(G), W(G)] \subseteq F(G)$ . On the other hand, we also show that there are interval cyclically colorable graphs for which feasible sets are not gap-free.

Our first two theorems give upper bounds for  $W_c(G)$  depending on the number of vertices and the maximum degree of the interval cyclically colorable graph  $G$ .

**Theorem 10** *If  $G$  is a connected triangle-free graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq |V(G)| + \Delta(G) - 2$ .*

**Proof.** Consider an interval cyclic  $W_c(G)$ -coloring  $\alpha$  of  $G$ . If for each  $u \in V(G)$ ,  $S(u, \alpha)$  is an interval of integers, then  $G \in \mathfrak{N}$  and  $W_c(G) \leq W(G) \leq |V(G)| - 1 \leq |V(G)| + \Delta(G) - 2$ , by Theorem 2 and taking into account that  $G$  is a connected triangle-free graph with at least two vertices.

Now suppose that there exists  $v_0 \in V(G)$  such that  $S(v_0, \alpha)$  is not an interval of integers. Since  $\alpha$  is an interval cyclic  $W_c(G)$ -coloring of  $G$ , for each  $v \in V(G)$  such that  $S(v, \alpha)$  is not an interval, there are colors  $k_v$  and  $l_v$  such that

$$S(v, \alpha) = \{1, \dots, k_v\} \cup \{W_c(G) - l_v + 1, \dots, W_c(G)\}.$$

Let  $l^* = \max_{\substack{v \in V(G), \\ S(v, \alpha) \text{ is not an interval}}} l_v$ . Clearly,  $1 \leq l^* \leq \Delta(G) - 1$ . Define an auxiliary graph  $H$  as follows:

$$\begin{aligned} V(H) &= V(G) \text{ and} \\ E(H) &= E(G) \setminus \{e : e \in E(G) \wedge \alpha(e) \in \{W_c(G) - l^* + 1, \dots, W_c(G)\}\}. \end{aligned}$$

Clearly,  $H$  is a spanning subgraph of  $G$ . Now let us consider the restriction of the coloring  $\alpha$  on the edges of subgraph  $H$  of  $G$ . Let  $\alpha_H$  be this edge-coloring. It is easy to see that  $\alpha_H$  is an interval  $(W_c(G) - l^*)$ -coloring of  $H$ . Moreover, since  $H$  is a triangle-free graph and  $H \in \mathfrak{N}$ , by Lemma 8, we have

$$W_c(G) - l^* \leq W(H) \leq |V(H)| - c(H) \leq |V(H)| - 1 = |V(G)| - 1.$$

This implies that  $W_c(G) \leq |V(G)| + l^* - 1$ . From this and taking into account that  $l^* \leq \Delta(G) - 1$ , we obtain  $W_c(G) \leq |V(G)| + \Delta(G) - 2$ .  $\square$

**Corollary 11** *If  $G$  is a connected triangle-free graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq 2|V(G)| - 3$ . Moreover, if  $G$  has at least three vertices, then  $W_c(G) \leq 2|V(G)| - 4$ .*

Note that the upper bound in Theorem 10 is sharp for simple cycles, since  $W_c(C_n) = n$ .

**Theorem 12** *If  $G$  is a connected graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq 2|V(G)| + \Delta(G) - 4$ . Moreover, if  $G$  has at least three vertices, then  $W_c(G) \leq 2|V(G)| + \Delta(G) - 5$ .*

**Proof.** Consider an interval cyclic  $W_c(G)$ -coloring  $\alpha$  of  $G$ . If for each  $u \in V(G)$ ,  $S(u, \alpha)$  is an interval of integers, then  $G \in \mathfrak{N}$  and  $W_c(G) \leq W(G) \leq 2|V(G)| - 3 \leq 2|V(G)| + \Delta(G) - 4$ , by Theorem 3 and taking into account that  $G$  is a connected graph with at least two vertices. Moreover, if  $G$  has at least three vertices, then  $W_c(G) \leq W(G) \leq 2|V(G)| - 4 \leq 2|V(G)| + \Delta(G) - 5$ , by Theorem 4.

Now suppose that there exists  $v_0 \in V(G)$  such that  $S(v_0, \alpha)$  is not an interval of integers. Since  $\alpha$  is an interval cyclic  $W_c(G)$ -coloring of  $G$ , for each  $v \in V(G)$  such that  $S(v, \alpha)$  is not an interval, there are colors  $k_v$  and  $l_v$  such that

$$S(v, \alpha) = \{1, \dots, k_v\} \cup \{W_c(G) - l_v + 1, \dots, W_c(G)\}.$$

Let  $l^* = \max_{\substack{v \in V(G), \\ S(v, \alpha) \text{ is not an interval}}} l_v$ . Clearly,  $1 \leq l^* \leq \Delta(G) - 1$ . Define an auxiliary graph  $H$  as follows:

$$\begin{aligned} V(H) &= V(G) \text{ and} \\ E(H) &= E(G) \setminus \{e : e \in E(G) \wedge \alpha(e) \in \{W_c(G) - l^* + 1, \dots, W_c(G)\}\}. \end{aligned}$$

Clearly,  $H$  is a spanning subgraph of  $G$ . Now let us consider the restriction of the coloring  $\alpha$  on the edges of the subgraph  $H$  of  $G$ . Let  $\alpha_H$  be this edge-coloring. It is easy to see that  $\alpha_H$  is an interval  $(W_c(G) - l^*)$ -coloring of  $H$ . Since  $H \in \mathfrak{N}$ , by Lemma 9, we have

$$W_c(G) - l^* \leq W(H) \leq 2|V(H)| - 3 \cdot c(H) \leq 2|V(H)| - 3 = 2|V(G)| - 3.$$

Moreover, if  $G$  has at least three vertices, then

$$W_c(G) - l^* \leq W(H) \leq 2|V(H)| - 4 \cdot c(H) \leq 2|V(H)| - 4 = 2|V(G)| - 4.$$

This implies that  $W_c(G) \leq 2|V(G)| + l^* - 3$ . From this and taking into account that  $l^* \leq \Delta(G) - 1$ , we obtain  $W_c(G) \leq 2|V(G)| + \Delta(G) - 4$ . Moreover, if  $G$  has at least three vertices, then  $W_c(G) \leq 2|V(G)| + \Delta(G) - 5$ .  $\square$

**Corollary 13** *If  $G$  is a connected graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq 3|V(G)| - 5$ . Moreover, if  $G$  has at least three vertices, then  $W_c(G) \leq 3|V(G)| - 6$ .*

Note that the first upper bound in Theorem 12 is sharp for  $K_2$  and the second upper bound is sharp for  $K_3$ , but we strongly believe that these upper bounds can be improved. Next we give some upper bounds for  $W_c(G)$  depending on degrees and diameter of the interval cyclically colorable connected graph  $G$ .

**Theorem 14** *If  $G$  is a connected graph and  $G \in \mathfrak{N}_c$ , then*

$$W_c(G) \leq 1 + 2 \cdot \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1),$$

where  $\mathbf{P}$  is the set of all shortest paths in the graph  $G$ .

**Proof.** Consider an interval cyclic  $W_c(G)$ -coloring  $\alpha$  of  $G$ . Let us show that  $W_c(G) \leq 1 + 2 \cdot \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1)$ . Suppose, to the contrary, that  $W_c(G) > 1 + 2 \cdot \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1)$ .

In the coloring  $\alpha$  of  $G$ , we consider the edges with colors 1 and  $2 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1)$ .

Let  $e = u_1 u_2$ ,  $e' = w_1 w_2$  and  $\alpha(e) = 1, \alpha(e') = 2 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1)$ . Without loss of generality we may assume that a shortest path  $P$  joining  $e$  with  $e'$  joins  $u_1$  with  $w_1$ , where

$$P = (v_0, e_1, v_1, \dots, v_{i-1}, e_i, v_i, \dots, v_{k-1}, e_k, v_k) \text{ and } v_0 = u_1, v_k = w_1.$$

Since  $\alpha$  is an interval cyclic coloring of  $G$ , we have

$$\begin{aligned} & \text{either } \alpha(e_1) \leq d_G(v_0) \text{ or } \alpha(e_1) \geq W_c(G) - d_G(v_0) + 2, \\ & \text{either } \alpha(e_2) \leq \alpha(e_1) + d_G(v_1) - 1 \text{ or } \alpha(e_2) \geq \alpha(e_1) - d_G(v_1) + 1, \\ & \dots \dots \dots \\ & \text{either } \alpha(e_i) \leq \alpha(e_{i-1}) + d_G(v_{i-1}) - 1 \text{ or } \alpha(e_i) \geq \alpha(e_{i-1}) - d_G(v_{i-1}) + 1, \\ & \dots \dots \dots \\ & \text{either } \alpha(e_k) \leq \alpha(e_{k-1}) + d_G(v_{k-1}) - 1 \text{ or } \alpha(e_k) \geq \alpha(e_{k-1}) - d_G(v_{k-1}) + 1. \end{aligned}$$

Summing up these inequalities, we obtain

$$\text{either } \alpha(e_k) \leq 1 + \sum_{j=0}^{k-1} (d_G(v_j) - 1) \text{ or } \alpha(e_k) \geq W_c(G) + 1 - \sum_{j=0}^{k-1} (d_G(v_j) - 1).$$

Hence, we have either

$$\alpha(e') \leq \alpha(e_k) + d_G(v_k) - 1 \leq 1 + \sum_{j=0}^k (d_G(v_j) - 1) \tag{1}$$

or

$$\alpha(e') \geq \alpha(e_k) - d_G(v_k) + 1 \geq W_c(G) + 1 - \sum_{j=0}^k (d_G(v_j) - 1). \tag{2}$$

On the other hand, by (1), we obtain

$$2 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1) = \alpha(e') \leq 1 + \sum_{j=0}^k (d_G(v_j) - 1) \leq 1 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1),$$

which is a contradiction.

Similarly, by (2), we obtain

$$2 + \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1) = \alpha(e') \geq W_c(G) + 1 - \sum_{j=0}^k (d_G(v_j) - 1) \geq W_c(G) + 1 - \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1) \text{ and thus } W_c(G) \leq 1 + 2 \cdot \max_{P \in \mathbf{P}} \sum_{v \in V(P)} (d_G(v) - 1), \text{ which is a contradiction. } \square$$

**Corollary 15** ([23]). *If  $G$  is a connected graph and  $G \in \mathfrak{N}_c$ , then*

$$W_c(G) \leq 1 + 2(\text{diam}(G) + 1)(\Delta(G) - 1).$$

**Theorem 16** *If  $G$  is a connected bipartite graph and  $G \in \mathfrak{N}_c$ , then*

$$W_c(G) \leq 1 + 2 \cdot \text{diam}(G)(\Delta(G) - 1).$$

**Proof.** Consider an interval cyclic  $W_c(G)$ -coloring  $\alpha$  of  $G$ . Let us show that  $W_c(G) \leq 1 + 2 \cdot \text{diam}(G)(\Delta(G) - 1)$ . Suppose, to the contrary, that  $W_c(G) > 1 + 2 \cdot \text{diam}(G)(\Delta(G) - 1)$ . In the coloring  $\alpha$  of  $G$ , we consider the edges with colors 1 and  $2 + \text{diam}(G)(\Delta(G) - 1)$ . Let  $e = u_1 u_2, e' = w_1 w_2$  and  $\alpha(e) = 1, \alpha(e') = 2 + \text{diam}(G)(\Delta(G) - 1)$ . Since for any two edges in a bipartite graph  $G$ , some two of their endpoints must be at a distance of at most  $\text{diam}(G) - 1$  from each other, we may assume that there is the path  $P$  joining  $e$  and  $e'$  with the length is not greater than  $\text{diam}(G) - 1$ . Also, we may assume that  $P$  joining  $e$  with  $e'$  joins  $u_1$  with  $w_1$ , where

$$P = (v_0, e_1, v_1, \dots, v_{i-1}, e_i, v_i, \dots, v_{k-1}, e_k, v_k) \text{ and } v_0 = u_1, v_k = w_1.$$

Since  $\alpha$  is an interval cyclic coloring of  $G$ , for  $1 \leq i \leq k$ , we have

$$\text{either } \alpha(v_{i-1}v_i) \leq 1 + \sum_{j=0}^{i-1} (d_G(v_j) - 1) \text{ or } \alpha(v_{i-1}v_i) \geq W_c(G) + 1 - \sum_{j=0}^{i-1} (d_G(v_j) - 1).$$

From this, we have either

$$\alpha(e') \leq 1 + \sum_{j=0}^k (d_G(v_j) - 1) \tag{3}$$

or

$$\alpha(e') \geq W_c(G) + 1 - \sum_{j=0}^k (d_G(v_j) - 1). \tag{4}$$

On the other hand, by (3) and taking into account that  $k \leq \text{diam}(G) - 1$ , we obtain

$$2 + \text{diam}(G)(\Delta(G) - 1) = \alpha(e') \leq 1 + \sum_{j=0}^k (d_G(v_j) - 1) \leq 1 + \text{diam}(G)(\Delta(G) - 1),$$

which is a contradiction.

Similarly, by (4) and taking into account that  $k \leq \text{diam}(G) - 1$ , we obtain

$2 + \text{diam}(G)(\Delta(G) - 1) = \alpha(e') \geq W_c(G) + 1 - \sum_{j=0}^k (d_G(v_j) - 1) \geq W_c(G) + 1 - \text{diam}(G)(\Delta(G) - 1)$  and thus  $W_c(G) \leq 1 + 2 \cdot \text{diam}(G)(\Delta(G) - 1)$ , which is a contradiction.  $\square$

Now we show that the coefficient 2 in the last upper bounds cannot be improved.



**Theorem 17** *For any integers  $d \geq 2$  and  $n \geq 3$ , there exists a connected graph  $G$  with  $\Delta(G) = d$  and  $\text{diam}(G) = \lfloor \frac{n}{2} \rfloor + 2$  such that  $G \in \mathfrak{N}_c$  and  $W_c(G) = n(d-1)$ .*

**Proof.** For the proof, we construct a graph  $G_{d,n}$  that satisfies the specified conditions. We define a graph  $G_{d,n}$  as follows:

$$V(G_{d,n}) = \{v_1, \dots, v_n\} \cup \left\{ u_j^{(i)} : 1 \leq i \leq n, 1 \leq j \leq d-2 \right\} \text{ and}$$

$$E(G_{d,n}) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1 v_n\} \cup \left\{ v_i u_j^{(i)} : 1 \leq i \leq n, 1 \leq j \leq d-2 \right\}.$$

Clearly,  $G_{d,n}$  is a connected graph with  $\Delta(G_{d,n}) = d$  and  $\text{diam}(G_{d,n}) = \lfloor \frac{n}{2} \rfloor + 2$ .

Let us show that  $G_{d,n}$  has an interval cyclic  $n(d-1)$ -coloring.

Define an edge-coloring  $\alpha$  of  $G_{d,n}$  as follows:

(1) for  $1 \leq i \leq n$  and  $1 \leq j \leq d-2$ , let

$$\alpha(v_i u_j^{(i)}) = (i-1)(d-1) + j;$$

(2) for  $1 \leq i \leq n-1$ , let

$$\alpha(v_i v_{i+1}) = i(d-1) \text{ and } \alpha(v_1 v_n) = n(d-1).$$

It is easy to see that  $\alpha$  is an interval cyclic  $n(d-1)$ -coloring of  $G_{d,n}$ . This implies that  $G_{d,n} \in \mathfrak{N}_c$  and  $W_c(G_{d,n}) \geq n(d-1)$ . On the other hand, clearly  $W_c(G_{d,n}) \leq |E(G_{d,n})| = n(d-1)$  and thus  $W_c(G_{d,n}) = n(d-1)$ .  $\square$

In the last part of the section we investigate the feasible sets of interval cyclically colorable graphs.

**Theorem 18** *If  $G \in \mathfrak{N}$ , then  $G \in \mathfrak{N}_c$  and  $[\Delta(G), W(G)] \subseteq F(G)$ .*

**Proof.** Since any interval  $t$ -coloring of  $G$  is also an interval cyclic  $t$ -coloring of  $G$ , we obtain that  $G \in \mathfrak{N}_c$ .

Assume that  $\Delta(G) \leq t \leq W(G)$ . Let  $\alpha$  be an interval  $W(G)$ -coloring of  $G$ . Define an edge-coloring  $\beta$  of  $G$  as follows: for every  $e \in E(G)$ , let

$$\beta(e) = \begin{cases} \alpha(e) \pmod{t}, & \text{if } \alpha(e) \pmod{t} \neq 0, \\ t, & \text{otherwise.} \end{cases}$$

It is easy to see that  $\beta$  is an interval cyclic  $t$ -coloring of  $G$ .  $\square$

**Corollary 19** ([23]). *If  $G \in \mathfrak{N}$ , then  $G \in \mathfrak{N}_c$  and  $w_c(G) = \Delta(G)$ .*

**Theorem 20** *If  $G$  is an Eulerian graph and  $|E(G)|$  is odd, then  $G$  has no interval cyclic  $t$ -coloring for every even positive integer  $t$ .*

**Proof.** Suppose, to the contrary, that  $G$  has an interval cyclic  $t$ -coloring  $\alpha$  for some even positive integer  $t$ . Since  $G$  is an Eulerian graph,  $G$  is connected and  $d_G(v)$  is even for any  $v \in V(G)$ , by Euler's Theorem. Since  $\alpha$  is an interval cyclic coloring and all degrees of vertices of  $G$  are even, we have that for any  $v \in V(G)$ , the set  $S(v, \alpha)$  contains exactly  $\frac{d_G(v)}{2}$  even colors and  $\frac{d_G(v)}{2}$  odd colors. Now let  $m_{\text{odd}}$  be the number of edges with odd colors in the coloring  $\alpha$ . By Handshaking lemma, we obtain  $m_{\text{odd}} = \frac{1}{2} \sum_{v \in V(G)} \frac{d_G(v)}{2} = \frac{|E(G)|}{2}$ .

Thus  $|E(G)|$  is even, which is a contradiction.  $\square$

**Corollary 21** *If  $G$  is an interval cyclically colorable Eulerian graph with an odd number of edges and  $\chi'(G) = \Delta(G)$ , then  $w_c(G) > \chi'(G)$ .*

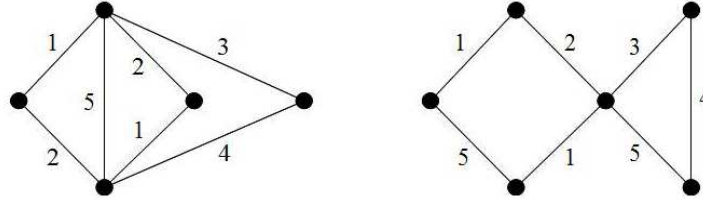


Figure 2. Interval cyclic 5-colorings of  $K_{1,1,3}$  and  $F$ .

Fig. 2 shows the complete tripartite graph  $K_{1,1,3}$  and the fish graph  $F$  that are smallest interval cyclically colorable Eulerian graphs with an odd number of edges and for which the chromatic index is equal to the maximum degree. Clearly,  $\chi'(K_{1,1,3}) = \chi'(F) = 4$ , but  $w_c(K_{1,1,3}) = w_c(F) = 5$ .

Finally let us consider the simple path  $P_m$  and the simple cycle  $C_n$  ( $n \geq 3$ ). Clearly,  $P_m \in \mathfrak{N}_c$  and  $w_c(P_m) = \Delta(P_m)$ ,  $W_c(P_m) = m - 1$ . Moreover,  $F(P_m) = [w_c(P_m), W_c(P_m)]$ , so the feasible set of  $P_m$  is gap-free. Now let us consider the simple cycle  $C_n$ . Clearly,  $C_n \in \mathfrak{N}_c$  and  $w_c(C_n) = \chi'(C_n)$ ,  $W_c(C_n) = n$ . Moreover, it is not difficult to see that any simple cycle with an odd number of vertices has an interval cyclic  $t$ -coloring for every odd integer  $t$ ,  $3 \leq t \leq n$ , so, by Theorem 20, we obtain

**Corollary 22** *For any odd integer  $n \geq 3$ , we have  $F(C_n) = [3, n]_{\text{odd}}$ .*

This corollary implies that for any odd integer  $n \geq 5$ , the feasible set of  $C_n$  is not gap-free. A more general result on the feasible set of simple cycles was obtained by Kamalian in [15].

**Theorem 23** ([15]). *For any integer  $n \geq 3$ , we have*

$$F(C_n) = \begin{cases} [3, n]_{\text{odd}}, & \text{if } n \text{ is odd,} \\ \left[2, \frac{n}{2} + 1\right] \cup \left[\frac{n}{2} + 2, n\right]_{\text{even}}, & \text{if } n = 4k, k \in \mathbb{N}, \\ \left[2, \frac{n}{2} + 1\right] \cup \left[\frac{n}{2} + 3, n\right]_{\text{even}}, & \text{if } n = 4k + 2, k \in \mathbb{N}. \end{cases}$$

#### 4. Interval cyclic edge-colorings of complete graphs

This section is devoted to interval cyclic colorings of complete graphs. In [31], Vizing proved the following

**Theorem 24** *For the complete graph  $K_n$  ( $n \geq 2$ ), we have*

$$\chi'(K_n) = \begin{cases} n - 1, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

From Corollary 6 and Theorem 24, we obtain that if  $n \in \mathbb{N}$ , then  $K_{2n}, K_{2n+1} \in \mathfrak{N}_c$  and  $w_c(K_{2n}) = 2n - 1$ ,  $w_c(K_{2n+1}) = 2n + 1$ . Now let us consider the parameters  $W_c(K_{2n})$  and  $W_c(K_{2n+1})$  when  $n \in \mathbb{N}$ . In [24], Petrosyan investigated interval colorings of complete graphs and hypercubes. In particular, he proved the following

**Theorem 25** *If  $n = p2^q$ , where  $p$  is odd and  $q$  is nonnegative, then*

$$W(K_{2n}) \geq 4n - 2 - p - q.$$

*Moreover, if  $2n - 1 \leq t \leq 4n - 2 - p - q$ , then  $K_{2n}$  has an interval  $t$ -coloring.*

**Corollary 26** *If  $n = p2^q$ , where  $p$  is odd and  $q$  is nonnegative, then*

$$W_c(K_{2n}) \geq 4n - 2 - p - q.$$

*Moreover,  $[2n - 1, 4n - 2 - p - q] \subseteq F(K_{2n})$ .*

On the other hand, Corollary 13 implies that  $W_c(K_{2n}) \leq 6n - 6$  for  $n \geq 2$ . It is not difficult to see that  $W(K_4) = W_c(K_4) = 4$ . In [24], it was proved that  $W(K_6) = 7$  and  $W(K_8) = 11$ , but Fig. 3 shows that  $W_c(K_6) \geq 9$  and  $W(K_8) \geq 12$ , so  $W_c(K_6) > W(K_6)$  and  $W_c(K_8) > W(K_8)$ . In general, we strongly believe that  $W_c(K_{2n}) > W(K_{2n})$  for  $n \geq 3$ . Now we give a lower bound for  $W_c(K_{2n+1})$  when  $n \in \mathbb{N}$ .

**Theorem 27** *If  $n \in \mathbb{N}$ , then  $W_c(K_{2n+1}) \geq 3n$ .*

**Proof.** For the proof, it suffices to construct an interval cyclic  $3n$ -coloring of  $K_{2n+1}$ . Let  $V(K_{2n+1}) = \{v_0, v_1, \dots, v_{2n}\}$ .

Define an edge-coloring  $\beta$  of  $K_{2n+1}$ . For each edge  $v_i v_j \in E(K_{2n+1})$  with  $i < j$ , define a color  $\beta(v_i v_j)$  as follows:

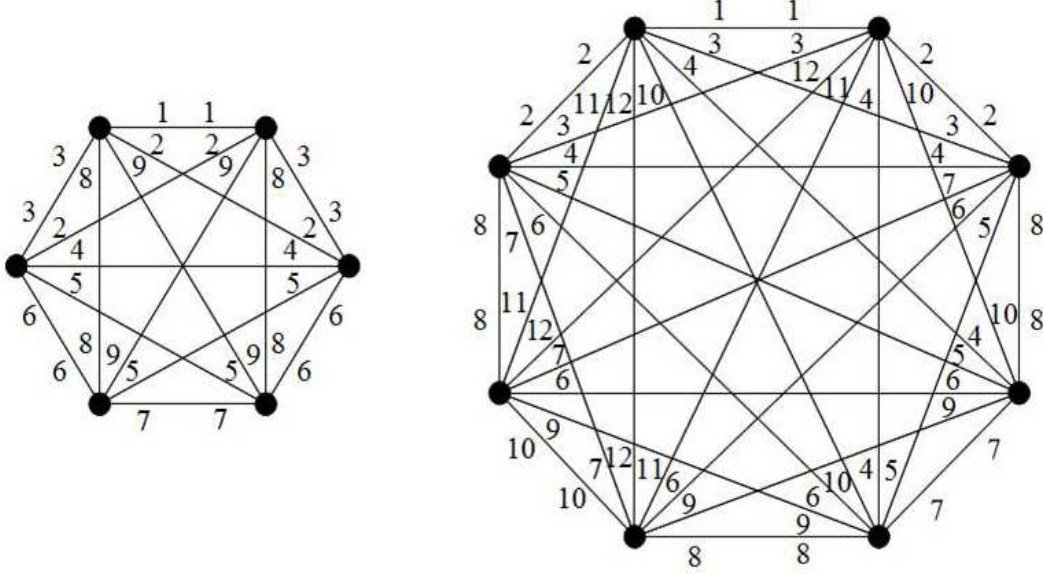


Figure 3. The interval cyclic 9-coloring of  $K_6$  and the interval cyclic 12-coloring of  $K_8$ .

$$\beta(v_i v_j) = \begin{cases} 1, & \text{if } i = 0, j = 1; \\ 2n + 1, & \text{if } i = 0, j = 2; \\ j - 1, & \text{if } i = 0, 3 \leq j \leq n; \\ n + 1 + j, & \text{if } i = 0, n + 1 \leq j \leq 2n - 2; \\ n, & \text{if } i = 0, j = 2n - 1; \\ 3n, & \text{if } i = 0, j = 2n; \\ i + j - 1, & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, 2 \leq j \leq n, i + j \leq n + 1; \\ i + j + n - 2, & \text{if } 2 \leq i \leq n - 1, \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n, i + j \geq n + 2; \\ n + 1 + j - i, & \text{if } 3 \leq i \leq n, n + 1 \leq j \leq 2n - 2, j - i \leq n - 2; \\ j - i + 1, & \text{if } 1 \leq i \leq n, n + 1 \leq j \leq 2n, j - i \geq n; \\ 2i - 1, & \text{if } 2 \leq i \leq 1 + \lfloor \frac{n-1}{2} \rfloor, n + 1 \leq j \leq n + \lfloor \frac{n-1}{2} \rfloor, j - i = n - 1; \\ i + j - 1, & \text{if } \lfloor \frac{n-1}{2} \rfloor + 2 \leq i \leq n, n + 1 + \lfloor \frac{n-1}{2} \rfloor \leq j \leq 2n - 1, j - i = n - 1; \\ i + j - 2n + 1, & \text{if } n + 1 \leq i \leq n + \lfloor \frac{n}{2} \rfloor - 1, n + 2 \leq j \leq 2n - 2, i + j \leq 3n - 1; \\ i + j - n, & \text{if } n + 1 \leq i \leq 2n - 1, n + \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq 2n, i + j \geq 3n. \end{cases}$$

Let us prove that  $\beta$  is an interval cyclic  $3n$ -coloring of  $K_{2n+1}$ .

Let  $G$  be the subgraph of  $K_{2n+1}$  induced by  $\{v_1, \dots, v_{2n}\}$ . Clearly,  $G$  is isomorphic to  $K_{2n}$ . This edge-coloring  $\beta$  of  $K_{2n+1}$  is constructed and based on the interval  $(3n - 2)$ -coloring of  $G$  which is described in the proof of Theorem 4 from [24]. We use this interval  $(3n - 2)$ -coloring of  $G$  and then we shift all colors of the edges of  $G$  by one. Let  $\alpha$  be this edge-coloring of  $G$ . Using the property of this edge-coloring which is described in the proof of Corollary 6 from [24], we get

- 1)  $S(v_1, \alpha) = S(v_2, \alpha) = [2, 2n]$ ,
- 2)  $S(v_i, \alpha) = S(v_{n+i-2}, \alpha) = [i, 2n - 2 + i]$  for  $3 \leq i \leq n$ ,
- 3)  $S(v_{2n-1}, \alpha) = S(v_{2n}, \alpha) = [n + 1, 3n - 1]$ .

Now, by the definition of  $\beta$ , we have

- 1)  $S(v_0, \beta) = [1, n] \cup [2n + 1, 3n]$ ,
- 2)  $S(v_1, \beta) = [1, 2n]$  and  $S(v_2, \beta) = [2, 2n + 1]$ ,
- 3)  $S(v_i, \beta) = [i - 1, 2n - 2 + i]$  and  $S(v_{n+i-2}, \beta) = [i, 2n - 1 + i]$  for  $3 \leq i \leq n$ ,
- 4)  $S(v_{2n-1}, \beta) = [n, 3n - 1]$  and  $S(v_{2n}, \beta) = [n + 1, 3n]$ .

This shows that  $\beta$  is an interval cyclic  $3n$ -coloring of  $K_{2n+1}$  and hence  $W_c(K_{2n+1}) \geq 3n$ .

□

Note that the lower bound in Theorem 27 is sharp for  $K_3$ , since  $W_c(K_3) = 3$ . On the other hand, Corollary 13 implies that  $W_c(K_{2n+1}) \leq 6n - 3$  for  $n \in \mathbb{N}$ . It is worth also noting that, in general, the feasible set of  $K_{2n+1}$  is not gap-free. For example,  $K_7$  has an interval cyclic 7-coloring and, by Theorem 27, it also has an interval cyclic 9-coloring, but since  $|E(K_7)| = 21$ , Theorem 20 implies that  $K_7$  has no interval cyclic 8-coloring, so  $F(K_7)$  is not gap-free.

## 5. Interval cyclic edge-colorings of complete bipartite and tripartite graphs

In this section we show that all complete bipartite and tripartite graphs are interval cyclically colorable. We also obtain some bounds for parameters  $W_c(K_{m,n})$  and  $w_c(K_{l,m,n})$  when  $l, m, n \in \mathbb{N}$ . In [12], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved the following

**Theorem 28** *For any  $m, n \in \mathbb{N}$ , we have*

- (1)  $K_{m,n} \in \mathfrak{N}$ ,
- (2)  $w(K_{m,n}) = m + n - \gcd(m, n)$ ,
- (3)  $W(K_{m,n}) = m + n - 1$ ,
- (4) *if  $w(K_{m,n}) \leq t \leq W(K_{m,n})$ , then  $K_{m,n}$  has an interval  $t$ -coloring.*

We first prove the theorem on the feasible set of complete bipartite graphs.

**Theorem 29** *If  $\min\{m, n\} = 1$ , then  $w_c(K_{m,n}) = W_c(K_{m,n}) = m + n - 1$ . If  $\min\{m, n\} \geq 2$ , then  $[\max\{m, n\}, m + n] \subseteq F(K_{m,n})$ .*

**Proof.** First note that if  $\min\{m, n\} = 1$ , then  $K_{m,n}$  is a star and hence  $w_c(K_{m,n}) = W_c(K_{m,n}) = m + n - 1$ .

Assume that  $\min\{m, n\} \geq 2$ . Let us show that if  $\max\{m, n\} \leq t \leq m+n$ , then  $K_{m,n}$  has an interval cyclic  $t$ -coloring. By Theorems 18 and 28, we have  $[\max\{m, n\}, m + n - 1] \subseteq F(K_{m,n})$ . Now we prove that  $K_{m,n}$  has an interval cyclic  $(m + n)$ -coloring.

Let  $V(K_{m,n}) = \{u_1, \dots, u_m, v_1, \dots, v_n\}$  and  $E(K_{m,n}) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ .

Define an edge-coloring  $\alpha$  of  $K_{m,n}$  as follows: for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , let

$$\alpha(u_i v_j) = \begin{cases} i + j - 1, & \text{if } (i, j) \neq (1, n), \\ m + n, & \text{otherwise.} \end{cases}$$

It is not difficult to see that  $\alpha$  is an interval cyclic  $(m + n)$ -coloring of  $K_{m,n}$  and hence  $[\max\{m, n\}, m + n] \subseteq F(K_{m,n})$  when  $\min\{m, n\} \geq 2$ .  $\square$

**Corollary 30** *If  $\min\{m, n\} = 1$ , then  $w_c(K_{m,n}) = W_c(K_{m,n}) = m + n - 1$ . If  $\min\{m, n\} \geq 2$ , then  $w_c(K_{m,n}) = \max\{m, n\}$  and  $W_c(K_{m,n}) \geq m + n$ .*

Now we show that all complete tripartite graphs are interval cyclically colorable.

**Theorem 31** *For any  $l, m, n \in \mathbb{N}$ , we have  $K_{l,m,n} \in \mathfrak{N}_c$  and  $w_c(K_{l,m,n}) \leq l + m + n$ .*

**Proof.** Without loss of generality we may assume that  $l \leq m \leq n$ . Clearly, for the proof, it suffices to construct an interval cyclic  $(l + m + n)$ -coloring of  $K_{l,m,n}$ .

Let  $V(K_{l,m,n}) = \{u_1, \dots, u_m, v_1, \dots, v_n, w_1, \dots, w_l\}$  and  $E(K_{l,m,n}) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i w_j : 1 \leq i \leq m, 1 \leq j \leq l\} \cup \{w_i v_j : 1 \leq i \leq l, 1 \leq j \leq n\}$ .

Define an edge-coloring  $\alpha$  of  $K_{l,m,n}$  as follows:

(1) for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , let

$$\alpha(u_i v_j) = l + i + j - 1;$$

(2) for  $1 \leq i \leq l$  and  $1 \leq j \leq n$ , let

$$\alpha(w_i v_j) = i + j - 1;$$

(3) for  $1 \leq i \leq m$ ,  $1 \leq j \leq l$  and  $i + j \leq m + 1$ , let

$$\alpha(u_i w_j) = l + n + i + j - 1;$$

(4) for  $1 \leq i \leq m$ ,  $1 \leq j \leq l$  and  $i + j \geq m + 2$ , let

$$\alpha(u_i w_j) = i + j - m - 1.$$

Let us prove that  $\alpha$  is an interval cyclic  $(l + m + n)$ -coloring of  $K_{l,m,n}$ .

By the definition of  $\alpha$ , we have

1) for  $1 \leq i \leq m - l + 1$ ,

$$S(u_i, \alpha) = [l + i, l + n + i - 1] \cup [l + n + i, 2l + n + i - 1] = [l + i, 2l + n + i - 1] \text{ due to (1) and (3),}$$

2) for  $m - l + 2 \leq i \leq m$ ,

$$S(u_i, \alpha) = [l + i, l + n + i - 1] \cup [l + n + i, l + m + n] \cup [1, l - m - 1 + i] = [1, l - m - 1 + i] \cup [l + i, l + m + n] \text{ due to (1), (3) and (4),}$$

3) for  $1 \leq i \leq n$ ,

$$S(v_i, \alpha) = [l + i, l + m + i - 1] \cup [i, l + i - 1] = [i, l + m + i - 1] \text{ due to (1) and (2),}$$

4) for  $1 \leq i \leq l$ ,

$$S(w_i, \alpha) = [i, n + i - 1] \cup [l + n + i, l + m + n] \cup [1, i - 1] = [1, n + i - 1] \cup [l + n + i, l + m + n] \text{ due to (2), (3) and (4).}$$

This implies that  $\alpha$  is an interval cyclic  $(l + m + n)$ -coloring of  $K_{l,m,n}$ ; thus  $K_{l,m,n} \in \mathfrak{N}_c$  and  $w_c(K_{l,m,n}) \leq l + m + n$  for  $l, m, n \in \mathbb{N}$ .  $\square$

**Corollary 32** *For any  $l, m, n \in \mathbb{N}$ , we have  $K_{l,m,n} \in \mathfrak{N}_c$  and  $W_c(K_{l,m,n}) \geq l + m + n$ .*

Note that the upper bound in Theorem 31 is sharp for  $K_{1,m,n}$  when  $m$  and  $n$  are odd, since  $w_c(K_{1,1,1}) = \chi'(C_3) = 3$  and for  $\max\{m, n\} \geq 3$ ,  $m + n + 1 \geq w_c(K_{1,m,n}) > \chi'(K_{1,m,n}) = \Delta(K_{1,m,n}) = m + n$  due to Corollary 21. It is worth also noting that although all complete tripartite graphs are interval cyclically colorable, in [9] Grzesik and Khachatryan proved that  $K_{1,m,n}$  is interval colorable if and only if  $\gcd(m + 1, n + 1) = 1$ . This implies that there are infinitely many complete tripartite graphs from the class  $\mathfrak{N}_c \setminus \mathfrak{N}$ .

## 6. Interval cyclic edge-colorings of hypercubes

In this section we show that hypercubes  $Q_n$  are interval cyclically colorable. We also obtain some bounds for the parameter  $W_c(Q_n)$  when  $n \in \mathbb{N}$ . In [24], Petrosyan investigated interval colorings of complete graphs and hypercubes. In particular, he proved that  $Q_n \in \mathfrak{N}$  and  $w(Q_n) = n$ ,  $W(Q_n) \geq \frac{n(n+1)}{2}$  for any  $n \in \mathbb{N}$ . In the same paper he also conjectured that  $W(Q_n) = \frac{n(n+1)}{2}$  for any  $n \in \mathbb{N}$ . In [26], the authors confirmed this conjecture. This implies that  $Q_n \in \mathfrak{N}_c$  and  $w_c(Q_n) = n$ ,  $W_c(Q_n) \geq \frac{n(n+1)}{2}$ . Moreover, by Theorem 18, we obtain  $\left[n, \frac{n(n+1)}{2}\right] \subseteq F(Q_n)$ . On the other hand, since  $Q_n$  is a connected bipartite graph and taking into account that  $\text{diam}(Q_n) = \Delta(Q_n) = n$ , we get  $W_c(Q_n) \leq 1 + 2 \cdot \text{diam}(Q_n) (\Delta(Q_n) - 1) = 2n^2 - 2n + 1$ , by Theorem 16. So, we have  $\frac{n(n+1)}{2} \leq W_c(Q_n) \leq 2n^2 - 2n + 1$  for any  $n \in \mathbb{N}$ . Now we prove a new lower bound for  $W_c(Q_n)$  which improves  $W_c(Q_n) \geq \frac{n(n+1)}{2}$  for  $2 \leq n \leq 5$ .

**Theorem 33** *For any integer  $n \geq 2$ , we have  $W_c(Q_n) \geq 4(n - 1)$ .*

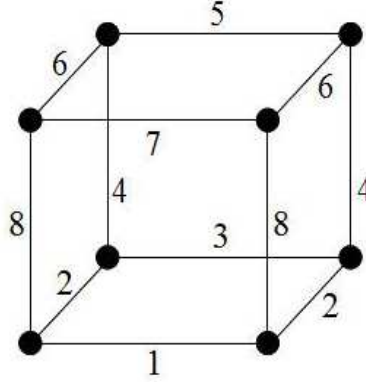


Figure 4. The interval cyclic 8-coloring of  $Q_3$ .

**Proof.** First let us note that for any integer  $n \geq 2$ ,  $Q_n$  has an interval  $(n+1)$ -coloring such that for one half of vertices of  $Q_n$ , the set of colors appearing on edges incident to these vertices is an interval  $[1, n]$  and for remaining half of vertices of  $Q_n$ , the set of colors appearing on edges incident to these remaining vertices is an interval  $[2, n+1]$ . It can be easily done by induction on  $n$ . Also, it is not difficult to see that  $W_c(Q_2) = W_c(C_4) = 4$  and  $W_c(Q_3) \geq 8$  (See Fig. 4).

Assume that  $n \geq 4$ .

For  $(i, j) \in \{0, 1\}^2$ , let  $Q_{n-2}^{(i,j)}$  be the subgraph of  $Q_n$  induced by the vertices

$$\{(i, j, \alpha_3, \dots, \alpha_n) : (\alpha_3, \dots, \alpha_n) \in \{0, 1\}^{n-2}\}.$$

Each  $Q_{n-2}^{(i,j)}$  is isomorphic to  $Q_{n-2}$ . Let  $\varphi$  be an interval  $(n-1)$ -coloring of  $Q_{n-2}^{(0,0)}$  such that for one half of vertices of  $Q_{n-2}^{(0,0)}$ , the set of colors appearing on edges incident to these vertices be an interval  $[1, n-2]$  and for remaining half of vertices of  $Q_{n-2}^{(0,0)}$ , the set of colors appearing on edges incident to these remaining vertices be an interval  $[2, n-1]$ .

Let us define an edge-coloring  $\psi$  of subgraphs  $Q_{n-2}^{(0,1)}$ ,  $Q_{n-2}^{(1,1)}$  and  $Q_{n-2}^{(1,0)}$  of  $Q_n$  as follows:

- (1) for every edge  $(0, 1, \bar{\alpha}) (0, 1, \bar{\beta}) \in E(Q_{n-2}^{(0,1)})$ , let

$$\psi((0, 1, \bar{\alpha}) (0, 1, \bar{\beta})) = \varphi((0, 0, \bar{\alpha}) (0, 0, \bar{\beta})) + n - 1;$$

- (2) for every edge  $(1, 1, \bar{\alpha}) (1, 1, \bar{\beta}) \in E(Q_{n-2}^{(1,1)})$ , let

$$\psi((1, 1, \bar{\alpha}) (1, 1, \bar{\beta})) = \varphi((0, 0, \bar{\alpha}) (0, 0, \bar{\beta})) + 2n - 2;$$

- (3) for every edge  $(1, 0, \bar{\alpha}) (1, 0, \bar{\beta}) \in E(Q_{n-2}^{(1,0)})$ , let



$$\psi((1, 0, \bar{\alpha})(1, 0, \bar{\beta})) = \varphi((0, 0, \bar{\alpha})(0, 0, \bar{\beta})) + 3n - 3.$$

Now we define an edge-coloring  $\lambda$  of the graph  $Q_n$ .

For every edge  $\tilde{\alpha}\tilde{\beta} \in E(Q_n)$ , let

$$\lambda(\tilde{\alpha}\tilde{\beta}) = \begin{cases} \varphi(\tilde{\alpha}\tilde{\beta}), & \text{if } \tilde{\alpha}, \tilde{\beta} \in V(Q_{n-2}^{(0,0)}), \\ \psi(\tilde{\alpha}\tilde{\beta}), & \text{if } \tilde{\alpha}, \tilde{\beta} \in V(Q_{n-2}^{(0,1)}) \text{ or } \tilde{\alpha}, \tilde{\beta} \in V(Q_{n-2}^{(1,1)}) \text{ or } \tilde{\alpha}, \tilde{\beta} \in V(Q_{n-2}^{(1,0)}), \\ n-1, & \text{if } \tilde{\alpha} \in V(Q_{n-2}^{(0,0)}), \tilde{\beta} \in V(Q_{n-2}^{(0,1)}), S(\tilde{\alpha}, \varphi) = [1, n-2], \\ n, & \text{if } \tilde{\alpha} \in V(Q_{n-2}^{(0,0)}), \tilde{\beta} \in V(Q_{n-2}^{(0,1)}), S(\tilde{\alpha}, \varphi) = [2, n-1], \\ 2n-2, & \text{if } \tilde{\alpha} \in V(Q_{n-2}^{(0,1)}), \tilde{\beta} \in V(Q_{n-2}^{(1,1)}), S(\tilde{\alpha}, \psi) = [n, 2n-3], \\ 2n-1, & \text{if } \tilde{\alpha} \in V(Q_{n-2}^{(0,1)}), \tilde{\beta} \in V(Q_{n-2}^{(1,1)}), S(\tilde{\alpha}, \psi) = [n+1, 2n-2], \\ 3n-3, & \text{if } \tilde{\alpha} \in V(Q_{n-2}^{(1,1)}), \tilde{\beta} \in V(Q_{n-2}^{(1,0)}), S(\tilde{\alpha}, \psi) = [2n-1, 3n-4], \\ 3n-2, & \text{if } \tilde{\alpha} \in V(Q_{n-2}^{(1,1)}), \tilde{\beta} \in V(Q_{n-2}^{(1,0)}), S(\tilde{\alpha}, \psi) = [2n, 3n-3], \\ 4n-4, & \text{if } \tilde{\alpha} \in V(Q_{n-2}^{(1,0)}), \tilde{\beta} \in V(Q_{n-2}^{(0,0)}), S(\tilde{\alpha}, \psi) = [3n-2, 4n-5], \\ 1, & \text{if } \tilde{\alpha} \in V(Q_{n-2}^{(1,0)}), \tilde{\beta} \in V(Q_{n-2}^{(0,0)}), S(\tilde{\alpha}, \psi) = [3n-1, 4n-4]. \end{cases}$$

It is easy to verify that  $\lambda$  is an interval cyclic  $(4n-4)$ -coloring of  $Q_n$ ; thus  $W_c(Q_n) \geq 4(n-1)$  for  $n \geq 2$ .  $\square$

This theorem implies that  $W_c(Q_2) = 4$ ,  $W_c(Q_3) \geq 8$ ,  $W_c(Q_4) \geq 12$  and  $W_c(Q_5) \geq 16$ . Moreover, it is not difficult to see that  $F(Q_2) = [2, 4]$ ,  $F(Q_3) = [3, 8]$ ,  $[4, 12] \subseteq F(Q_4)$  and  $[5, 16] \subseteq F(Q_5)$ . We strongly believe that the feasible set of  $Q_n$  is gap-free, but this is an open problem.

## 7. Graphs that have no interval cyclic edge-coloring

In this section we describe two methods for constructing of interval cyclically non-colorable graphs. Our first method is based on trees and it was earlier used for constructing of interval non-colorable graphs in [28].

Let  $T$  be a tree and  $V(T) = \{v_1, \dots, v_n\}$ ,  $n \geq 2$ . Let  $P(v_i, v_j)$  be a simple path joining  $v_i$  and  $v_j$  in  $T$ ,  $VP(v_i, v_j)$  and  $EP(v_i, v_j)$  denote the sets of vertices and edges of this path, respectively. Also, let  $L(T) = \{v : v \in V(T) \wedge d_T(v) = 1\}$ . For a simple path  $P(v_i, v_j)$ , define  $LP(v_i, v_j)$  as follows:

$$LP(v_i, v_j) = |EP(v_i, v_j)| + |\{vw : vw \in E(T), v \in VP(v_i, v_j), w \notin VP(v_i, v_j)\}|.$$

Define:  $M(T) = \max_{1 \leq i < j \leq n} LP(v_i, v_j)$ . Let us define the graph  $\tilde{T}$  as follows:

$$V(\tilde{T}) = V(T) \cup \{u\}, u \notin V(T), E(\tilde{T}) = E(T) \cup \{uv : v \in L(T)\}.$$

Clearly,  $\tilde{T}$  is a connected graph with  $\Delta(\tilde{T}) = |L(T)|$ . Moreover, if  $T$  is a tree in which the distance between any two pendant vertices is even, then  $\tilde{T}$  is a connected bipartite graph.

In [14], Kamalian proved the following result.

**Theorem 34** *If  $T$  is a tree, then*

- (1)  $T \in \mathfrak{N}_c$ ,
- (2)  $w_c(T) = \Delta(G)$ ,
- (3)  $W_c(T) = M(T)$ ,
- (4)  $F(T) = [w_c(T), W_c(T)]$ .

**Theorem 35** *If  $T$  is a tree and  $|L(T)| \geq 2(M(T) + 2)$ , then  $\tilde{T} \notin \mathfrak{N}_c$ .*

**Proof.** Suppose, to the contrary, that  $\tilde{T}$  has an interval cyclic  $t$ -coloring  $\alpha$  for some  $t \geq |L(T)|$ .

Consider the vertex  $u$ . Without loss of generality we may assume that  $S(u, \alpha) = [1, |L(T)|]$ . Let  $v$  and  $v'$  be two vertices adjacent to  $u$  such that  $\alpha(uv) = 1$  and  $\alpha(uv') = M(T) + 3$ . Since  $\tilde{T} - u$  is a tree, there is a unique path  $P(v, v')$  in  $\tilde{T} - u$  joining  $v$  with  $v'$ , where

$$P(v, v') = (x_0, e_1, x_1, \dots, x_{i-1}, e_i, x_i, \dots, x_{k-1}, e_k, x_k), \quad x_0 = v, \quad x_k = v'.$$

Since  $\alpha$  is an interval cyclic coloring of  $G$ , for  $1 \leq i \leq k$ , we have

$$\text{either } \alpha(x_{i-1}x_i) \leq 2 + \sum_{j=0}^{i-1} (d_T(x_j) - 1) \text{ or } \alpha(x_{i-1}x_i) \geq t - \sum_{j=0}^{i-1} (d_T(x_j) - 1).$$

From this, we have either

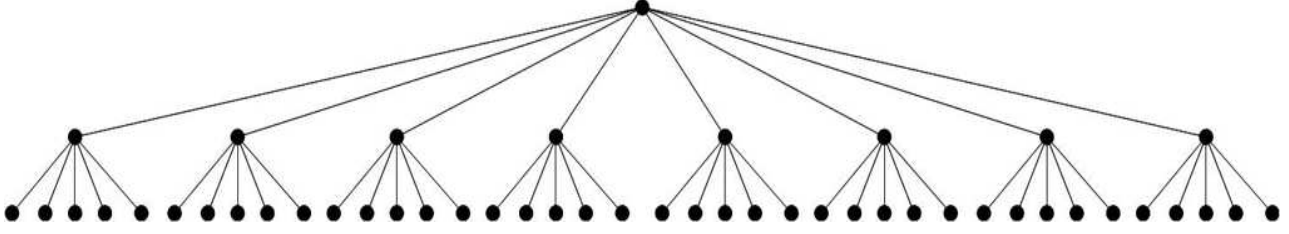
$$\alpha(x_{k-1}x_k) = \alpha(x_{k-1}v') \leq 2 + \sum_{j=0}^{k-1} (d_T(v_j) - 1) = 1 + LP(v, v') \leq 1 + M(T) \quad (5)$$

or

$$\alpha(x_{k-1}x_k) = \alpha(x_{k-1}v') \geq t - \sum_{j=0}^{k-1} (d_T(v_j) - 1) = t + 1 - LP(v, v') \geq t + 1 - M(T). \quad (6)$$

On the other hand, by (5), we obtain  $M(T) + 3 = \alpha(uv') \leq 2 + M(T)$ , which is a contradiction. Similarly, by (6), we obtain  $M(T) + 3 = \alpha(uv') \geq t - M(T)$  and thus  $t \leq 2M(T) + 3$ , which is a contradiction.  $\square$

**Corollary 36** *If  $T$  is a tree in which the distance between any two pendant vertices is even and  $|L(T)| \geq 2(M(T) + 2)$ , then the bipartite graph  $\tilde{T}$  has no interval cyclic coloring.*

Figure 5. The tree  $T$ .

Now let us consider the tree  $T$  shown in Fig. 5. Since  $M(T) = 18$  and  $|L(T)| = 40$ , the bipartite graph  $\tilde{T}$  with  $|V(\tilde{T})| = 50$  and  $\Delta(\tilde{T}) = 40$  has no interval cyclic coloring.

The second method which we consider is based on complete graphs and it was first described in [23], but here we prove a more stronger result.

Let  $K_{2n+1}$  be a complete graph on  $2n+1$  vertices and  $V(K_{2n+1}) = \{v_1, \dots, v_{2n+1}\}$ . For any  $m, n \in \mathbb{N}$ , define the graph  $K_{2n+1}^{\star m}$  as follows:

$$\begin{aligned} V(K_{2n+1}^{\star m}) &= V(K_{2n+1}) \cup \{u, w_1, \dots, w_m\}, \\ E(K_{2n+1}^{\star m}) &= E(K_{2n+1}) \cup \{v_1 u\} \cup \{uw_i : 1 \leq i \leq m\}. \end{aligned}$$

Clearly,  $K_{2n+1}^{\star m}$  is a connected with  $|V(K_{2n+1}^{\star m})| = m + 2n + 2$  and  $\Delta(K_{2n+1}^{\star m}) = \max\{m + 1, 2n + 1\}$ .

**Theorem 37** *If  $n \geq 2$  and  $m \geq 6n$ , then  $K_{2n+1}^{\star m} \notin \mathfrak{N}_c$ .*

**Proof.** Suppose, to the contrary, that  $K_{2n+1}^{\star m}$  has an interval cyclic  $t$ -coloring  $\alpha$  for some  $t \geq d(u) = 6n + 1$ .

Let  $H = K_{2n+1}^{\star m} - w_1 - w_2 - \dots - w_m$ . Also, let  $C = \bigcup_{v \in V(H)} S(v, \alpha)$  and  $\overline{C} = [1, t] \setminus C$ . Since  $H$  is connected, it is not difficult to see that either  $C$  or  $\overline{C}$  is an interval of integers. Let  $|C| = t'$ . Clearly,  $t' \leq t$ . Now let us consider the restriction of the coloring  $\alpha$  on the edges of the subgraph  $H$  of  $K_{2n+1}^{\star m}$ . Let  $\alpha_H$  be this edge-coloring. By rotating of colors of  $C$  along the cycle with colors  $1, \dots, t$ , we get a new edge-coloring  $\alpha'_H$  of  $H$  with colors  $1, \dots, t'$ . Since  $\alpha$  is an interval cyclic  $t$ -coloring of  $K_{2n+1}^{\star m}$  and taking into account that the vertex  $u$  in  $H$  is pendant, we obtain that  $\alpha'_H$  is an interval cyclic  $t'$ -coloring of  $H$ . Moreover, by Corollary 13, we have  $t' \leq 3|V(H)| - 6 = 3(2n + 2) - 6 = 6n$ . Since  $t \geq 6n + 1$ , we get that  $\alpha'_H$  is also an interval  $t'$ -coloring of  $H$ . In [8], it was proved that  $H \notin \mathfrak{N}$ , so this contradiction proves the theorem.  $\square$

**Corollary 38** *For any integer  $d \geq 13$ , there exists a connected graph  $G$  such that  $G \notin \mathfrak{N}_c$  and  $\Delta(G) = d$ .*

Now we show that  $K_5^{\star 11} \notin \mathfrak{N}_c$ . Note that  $|V(K_5^{\star 11})| = 17$  and  $\Delta(K_5^{\star 11}) = 12$ . Suppose that, to the contrary, that  $K_5^{\star 11}$  has an interval cyclic  $t$ -coloring  $\alpha$  for some  $t \geq 12$ . Similarly as in the proof of Theorem 37, we can consider the subgraph  $H = K_5^{\star 11} - w_1 -$

$w_2 - \dots - w_{11}$  of  $K_5^{*11}$ . Let  $t' = \left| \bigcup_{v \in V(H)} S(v, \alpha) \right|$  and  $\alpha_H$  be the restriction of the coloring  $\alpha$  on the edges of the subgraph  $H$  of  $K_5^{*11}$ . Then, let  $\alpha'_H$  be the edge-coloring of  $H$  with colors  $1, \dots, t'$ . Since  $\alpha$  is an interval cyclic  $t$ -coloring of  $K_5^{*11}$  and taking into account that the vertex  $u$  in  $H$  is pendant, it can be easily seen that  $\alpha'_H$  is an interval cyclic  $t'$ -coloring of  $H$ , where  $t' \leq t$ . Clearly,  $t' \leq |E(H)| = 11$ . Since  $t \geq 12$ , we obtain that  $\alpha'_H$  is also an interval  $t'$ -coloring of  $H$ , which is a contradiction. From here, we get the following

**Corollary 39** *For any integer  $d \geq 12$ , there exists a connected graph  $G$  such that  $G \notin \mathfrak{N}_c$  and  $\Delta(G) = d$ .*

## 8. Problems and Conjectures

In this section we collected different problems and conjectures that arose in previous sections. Our first conjectures concern the parameters  $w_c(G)$  and  $W_c(G)$  of an interval cyclically colorable graph  $G$ . In section 3, we proved that if  $G$  is a connected triangle-free graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq |V(G)| + \Delta(G) - 2$ . However, we think that the maximum degree in the upper bound can be omitted; more precisely we believe that the following is true:

**Conjecture 40** *If  $G$  is a connected triangle-free graph and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq |V(G)|$ .*

Note that if Conjecture 40 is true, then this upper bound cannot be improved, since  $W_c(K_{m,n}) \geq m + n$  ( $\min\{m, n\} \geq 2$ ), by Corollary 30. We also proved that if  $G$  is a connected graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq 2|V(G)| + \Delta(G) - 4$ . We again think that the maximum degree in this upper bound can be omitted; more precisely we believe that the following is true:

**Conjecture 41** *If  $G$  is a connected graph with at least two vertices and  $G \in \mathfrak{N}_c$ , then  $W_c(G) \leq 2|V(G)| - 3$ .*

It is worth noting that there exists a connection between Conjecture 40 and Conjecture 41. If one can prove that Conjecture 40 is true, then we are able to show that  $W_c(G) \leq 2|V(G)| - 1$  for an interval cyclically colorable connected graph  $G$ .

It is known that all regular graphs  $G$  are interval cyclically colorable and  $w_c(G) = \chi'(G)$ . Moreover, if  $G$  is interval colorable, then  $G$  is interval cyclically colorable and  $w_c(G) = \chi'(G) = \Delta(G)$ . On the other hand, in section 3, it was shown that there are many interval cyclically colorable graphs  $G$  for which  $w_c(G) > \chi'(G)$ . So, it is interesting to investigate the following

**Problem 1** *Characterize all interval cyclically colorable graphs  $G$  for which  $w_c(G) = \chi'(G)$ .*

In section 3, we also investigated the feasible sets of interval cyclically colorable graphs. In particular, we proved that if  $G$  is interval colorable, then  $[\Delta(G), W(G)] \subseteq F(G)$ . On the other hand, we gave some examples of interval cyclically graphs  $G$  for which  $F(G)$  is not gap-free. So, it is interesting to investigate the following

**Problem 2** *Characterize all interval cyclically colorable graphs  $G$  for which  $F(G)$  is gap-free.*

For example, we know that if  $T$  is a tree, then  $F(T)$  is gap-free [14], but we also strongly believe that for any  $m, n \in \mathbb{N}$ ,  $F(K_{2n})$ ,  $F(K_{m,n})$  and  $F(Q_n)$  are gap-free.

In sections 4 and 5, we investigated interval cyclic colorings of complete, complete bipartite and tripartite graphs, but the following problems are still open:

**Problem 3** *What is the exact value of  $W_c(K_n)$  for any  $n \in \mathbb{N}$ ?*

**Problem 4** *What are the exact values of  $w_c(K_{l,m,n})$  and  $W_c(K_{m,n})$ ,  $W_c(K_{l,m,n})$  for any  $l, m, n \in \mathbb{N}$ ?*

In sections 4 and 5, we proved that all complete bipartite and tripartite graphs are interval cyclically colorable, but we think that a more general result is true:

**Conjecture 42** *All complete multipartite graphs are interval cyclically colorable.*

In section 6, we investigated interval cyclic colorings of hypercubes  $Q_n$  and proved that  $W_c(Q_n) = O(n^2)$ , but the following problem remains open:

**Problem 5** *What is the exact value of  $W_c(Q_n)$  for any  $n \in \mathbb{N}$ ?*

In [23], Nadolski showed that if  $G$  is a connected graph with  $\Delta(G) = 3$ , then  $G \in \mathfrak{N}_c$  and  $w_c(G) \leq 4$ . From here and taking into account that all simple paths and cycles are interval cyclically colorable, we obtain that all subcubic graphs are interval cyclically colorable. On the other hand, in section 7, we proved that for any integer  $d \geq 12$ , there exists a connected graph  $G$  such that  $G \notin \mathfrak{N}_c$  and  $\Delta(G) = d$ . So, it is naturally to consider the following

**Problem 6** *Is there a connected graph  $G$  such that  $4 \leq \Delta(G) \leq 11$  and  $G \notin \mathfrak{N}_c$ ?*

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